Why and how to implement a digital measuring tool that supports the rotational angle aspect

Daniel Milow
E-mail: milow@dmilow.de
Universität Potsdam
Potsdam, Brandenburg, Germany

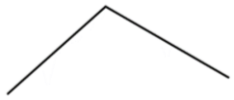
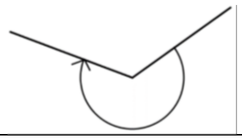
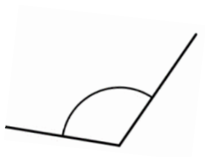
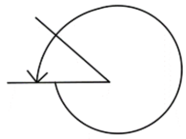
Abstract

Studies have shown that student's angle "Grundvorstellungen"(basic ideas) often do not reflect the definitions of angles properly and only some of the possible views on angles are being held. Students develop a sustaining misconception of angle in relation to Euclidean distance, an idea which is even supported by most traditional measuring instruments in Germany. In close connection to this it can be shown, that a revision of already existing protractors is inevitable for the development of an appropriate angle understanding. Dynamic views on angles like rotation and turning propose a solution for this problem but are rarely integrated in German math lessons or current textbooks. An IGS (interactive geometry software) and its dynamic approach in particular can be utilized to support a deeper understanding of the angle concept with respect to the rotational angle aspect. Thus, it should be integrated into an IGS angle measurement tool. To reach this goal, mathematical, didactical and design aspects of an IGS are being analysed for angle measurements. As a result, a tangible tool for angular measurements related to the rotational aspect of the angle concept has been developed. Since the development of digital geometrical tools takes place in the area between mathematics and its didactic, computer science and cognitive psychology, one must simultaneously consider the implementation in all other disciplines. Therefore, the development of such a tool is not a linear process but rather an iterative one after developing the presumptions. A starting point for this article should be the mathematical definition of the term angle which directed by the mathematic-didactical circumstances supports a first concept of the angular measurement tool. Only then should design decisions contribute to the completion.

1. Introduction: definition of the term angle

When asking mathematicians for a definition of the term angle one presumably gets as many different answers as there are mathematicians. One possible starting point for analyzing the term can be the measurement. Depending on how and for what the angle is measured defines whether or not an angle is strictly positive and in which area and unit it is measured. Therefore, if the angle and its measure are being analyzed further as mathematical terms, the understanding behind it needs to be clarified first. A truly comprehensive summary of different possible angle concepts can be found in the work of Krainer [1]. He not only subsumed the inner mathematical views but also the mathematic-didactical one which is why the work of Krainer is of extraordinary significance. In his meta-analyses, he aims to summarize all possible angle-definitions within only a few categories. Besides the definitions of mathematicians, Krainer also analyzed typical mathematical tasks in order to find truly meaningful categories. It is noteworthy that due to a suggested active movement two of his categories show a dynamic character while the other two remain more static in their configuration (Table 1). Nevertheless, there are attempts to find a cohesive definition. For instance, Etzold argues that different angle-definitions are useful and necessary but that there is also a need for a generalized concept which makes a comparison of different angle-situations on a mutual basis possible to begin with. Deriving from that he proposes to describe the angle as a ray with a corresponding size measurement and to define this understanding as "informational angle" [2]. At least all of Krainers condensed definitions can be summarized conceptionally under this banner, which implies a commonality of the proposition for education. This approach can certainly be useful as a starting point for the definition of the term angle but in order to practical measure an angle; a more distinct and practical one is needed. On the contrary, it seems that many different definitions also warrant their existence even if they are difficult to covenant among each other.

Table 1: *Categories of angles in [2]*

More static	More dynamic
angles without arc 	angles with arc-arrow / oriented angle-field 
angles with arc/angle-field 	angles with revolution-arrow 

But no matter how mathematicians define angles within their field, the situation in mathematics classes tends to differ. Pedagogical and field-pedagogical circumstances always need to be addressed when learning and teaching mathematical terms. From the described complexity and abstractness of the term angle, false ideas can easily arise if students generalize the wrong aspects of exemplary situations. The creation of the term angle within the imagination of learners can be viewed as a form of paradigm shift that also depends on epistemological and psychosocial aspects. Precisely because they know angle situations from their everyday lives, in mathematics, they are confronted with theoretical terms and concepts that compete with their everyday experiences. Therefore, the product of this conflict between two worlds can be a formally false understanding of the subject area "angle". But since the false understanding derives from the shared everyday life of the students, typical misconceptions arise which the tool presented in this work shall address. We know some of these typical misconceptions from several surveys explained below. That said students often don't know what is actually measured via angles and the angle measure and so in a logical consequence imagine that the length of the line segment equals the angle measure. Subsequently, because of the lack of a different concept, the angle is understood in terms of length or distance. By contrast, a number of learners also interprets the size of the angle mark itself as the measurement for the angle ([3] after [1]). This misinterpretation of different dimensions becomes particularly apparent when students try to define what 1° is. In this task, this special angular measure often gets identified with the distance of the sides. When confronted with this question students often think that the sides of a 1° angle have a distance of exactly one millimeter ([4], [5]). This attribution of a linear distance measurement to the angle could be critically questioned through a rotationally based term. If the angle term is viewed as a result of a rotation (periodically or not), the distance measurement becomes meaningless. With this in mind, the definition of angles from Freudenthals perspective can be helpful. With his analytical angles based on rotation, he defined an angular measure that is neither restricted nor periodical [6]. One of his concrete reasoning for this definition is that the rotation of a key for a half turn is simply not the same as for one and a half turn. A comparison between an acute and its complementing reflex angle at the latest should ignite a helpful conflict in order to eliminate the misconception of students ([7], [8]). It seems that the rotation as a dynamic activation of the angle, despite being described as one of the most natural [9], has not played a major role in German mathematics classes so far. Students in general mainly perceive angular situations in a static way while a more dynamic view is adapted only on rare occasions ([4], [10], [11]). However, how could they adapt to a more dynamic perception when most German educational books do not even present a dynamic point of view for measuring angles? [12] For a holistic angular concept, a diverse and situational view, which also includes the rotational aspect, seems appropriate. Therefore, Krainer particularly describes his environmental and application-based tuition with the identification of defining attributes of various angular terms ([1], p.32). It is clear that many of these problematic assumptions can only be adequately dealt with in good and vivid mathematics classes. So the question needs to be what a dynamic geometrical platform (IGS) independent from classes can offer in order to support a versatile but above all correct angular concept. Why it generally makes sense to use IGS in mathematical classes shall not be discussed in further detail here, since other works have already done this very clearly (e.g. in [13]). Here it shall only be explained how IGS can be of tangible help for a deeper understanding of the angular concept and for angular measurement. For this, the basic understanding on angles shall be picked up conceptionally.

In order to differentiate and characterize the individual student concepts in mathematics the term of the basic mathematical competencies ("Grundvorstellungen") was especially characterized by Vom Hofe [14]. In the descriptive interpretation, his assumptions consist of three partial aspects. If the transfer capacity of only one of these partial aspects cannot be adduced, typical misconceptions arise. The connection to known correlations (first aspect of the basic mathematical competencies) is clearly determined by the conceptional thought of a tool. If it adequately represents reality in its functionality, students will be aided in their conceptual genesis. If it complements the existing view of the world the student can develop a more complete angular term by adding new aspects. Moreover, the digital environment may also address angular aspects which cannot be attained by regular analog tools or which are reflected impractically (e.g. rotations and overrotation). Formal visual representations (second aspect of the basic mathematical competencies) are being implemented primarily in classes but can and have to be coherent to geometrical programs. However, this aspect of basic assumptions offers fewer chances for tools in IGS but instead formulates additional requirements. Finally, the third aspect of the basic mathematical competencies, the ability to apply a term is also part of a geometric program: If students learn how to apply a tool for angular measurement they can usually apply at least one concept of the angle. Therefore, tools for angular measurement in IGS can also contribute to the development of a correct understanding of the terms.

2. Research Questions

The research questions for this article subsequently arise from the current state of research concerning student's comprehension of the angular term in view of the chances of modern dynamic geometric programs. We know different perspectives on the term-construction of the angle from the mathematical and mathematic-didactical point of view. We also know typical problems which result from insisting on a particular concept of learning. Furthermore, it seems like the underrepresented dynamic term-definitions present a solution for some of these problems and therefore, should find a place in geometric programs. Thus, the following questions need to be answered for the development of a helpful angular measurement tool:

1. How should an angle measurement tool behave in a dynamic geometry environment, so that the rotational aspect of the angle is so prominent that typical misconceptions of learners can be addressed?
2. How should this tool be used to ensure both adequate functionality as well as intuitive usability?

In order to even answer these questions, already existing analog and digital tools shall be briefly analyzed. Furthermore, it will be discussed which functions are necessary and useful for the application within a school context. The practical part of the article will serve to implement the developed theoretical tool; among other things by commenting on which mathematical problems may occur and how the user input should best be embedded.

3. Learning the angular concept with a digital measurement tool

As already stated before generating the research questions the rotational aspect of the angle is usually underrepresented despite being potentially helpful. Therefore, it would make sense for a new angular tool to include this aspect. Moreover, it should be borne in mind that students need to be able to measure, even though they maybe internalized another angle aspect. However, it makes no sense to provide a separate tool for every aspect of the angle. Krainer noticed that math lessons, even though a diverse angle definition may be helpful, should not pursue a complex “science of angles” (“Winkologie” in [1]). This also needs to be assumed in the context of digital protractors (even if some design considerations yield the same result). Thus, if possible a tool needs to combine several perspectives or at the least don't stand in conflict with them. The advantage of a tool in contrast to the concrete tuition content is that theoretical background knowledge can actually stay in the background. A mathematical tool does not need to explain how and on what basis it works. Only handling and outcome play a role for the user. The result is that the actual different aspects of the angle can be pictured to a great extent by the same tool. In order to even realize such a common measurement approach the foundation needs to be standardized. Furthermore, the choice of the angle-creating objects in some cases already determines an angular definition and among other things limits the measuring range. But in order to not already establish one single definition, an angle should be measured with the designation of three points. This specifies a definition of the second grade: not the actual angle-creating objects are stated but the objects that beforehand created them. This can initially include every definition based on line segments, rays and straights as well as turn and rotational aspects. It is clear that not all terms can be considered equally. A switch of the angular field while turning a straight line beyond the right angle, for example, is only intended with the definition of not ordered straight lines. In order to avoid that the false angle is being measured the illustration through angular markers is now mandatory. With this marking a user can see which angle is being measured right now and compare them with his own imagination. If both do not add up, the marking can be redefined and measured again. However, this circumstance does not present a problem since the selection of the angle to be measured is also necessary for manual angular measurement. An advantage of the dynamic of geometrical systems is the possibility to clearly observe angles now that were previously ambiguously on the static plane. It is merely necessary to ensure a form of tracing of the user input. Therefore, it is theoretically possible to record over rotation. However, there is the problem of an appropriate angular marking. Spirals, which are often used for this can quickly become confusing resulting in a marking that doesn't illustrate the measured angle. Then there is also the question of whether or not continuous rotations are useful. In which context do we need this angular concept? What can this aspect further add to a complex angular understanding that is not already provided by periodical rotations? Since this article cannot answer these questions, two varieties of the tool shall be offered: One with implemented overrotations and one without them.

Additional features of the more dynamic definition are the distinction of an orientation as well as their consideration within the angular marking. Having said that a change of the algebraic sign of the angular measure can be confusing for students without an oriented imagination. Therefore, left and right rotations should be marked separately next to the angular measure. If students deem this information to be necessary it is there, but it is not mandatory to understand in order to interpret the angular measure. A similar thought is the distinction between the directions of the marking. Rotational based views usually add an arrow to circular arcs and spirals in order to distinguish the original and the turned side. However, this step would also prefer an explicit point of view which is why another possibility should be developed. The idea here is to start from the static angular marking and to illustrate the first side with the help of a gradual transparency of the filling. This shouldn't confuse static imagination but still, enable orientation. It should also be noted that this is a mere working hypothesis and that there is no research known to the author concerning gradually marked angles.

4. Experiences from existing tools

From mathematical classes in Germany, most students are familiar with the “Geodreieck” as the tool for angular measurement (a set square with additional projected angle scale on its sides) but half and full angular protractors are also accounted for in textbooks ([1], [15]). Measuring tools like the Goniometer, which are primarily based on the rotational aspect, on the other side are far less known in our country and are exclusively used in certain professional branches. This still applies today despite the potential help of the rotational concept for a paradigm shift from every day to the mathematical imagination. In order to develop a tool for this work, a few existing tools will be looked at closer.

Regarding analog tools

The Geodreieck is primarily used to measure static angles, even if a dynamic and rotational based approach is possible [16]. For this tool, Krainer assumes that the scaling with lines at its sides is confusing and deepens the imagination of distance of students [1]. That said it is actually not easier to notice the difference to a linear scale with a marking of dots. Nonetheless, the Geodreieck is a multi-functional tool – with its angle and distances measurement functionality it has proven itself within German-speaking countries.

In contrast to the Geodreieck, half and full circle protractors have the advantage of supporting misconceptions of length measures in a lesser way. Instead, the form of the tools indicates a closer connection to concentric circles that cannot be found in the design of the Geodreieck. Krainer also supports this focus on the origin of the angular dimension and even views the active construction of a half or full circle protractor as a meaningful occupation [1]. Also due to its regularity, a full circle protractor supports a dynamic angle measurement more than the Geodreieck. In general, when measuring with circle-based protractors it is often difficult for students to attach the leg of the tool to the leg of the angle [3].

Regarding digital tools

In their standard configuration most of the IGS, which enable angular measurements, promote a definition of the angle which is based on a sorted pair of rays, thus they are goniometric. Only Cinderella is based on a different approach. Here the angular measurement and the angular marking work completely detached from each other: while the measuring works analytical, thus angles beyond the full angle are possible, the angular marking is motivated by elementary geometry (based on a non-ordered pair of half-lines). However, the result is a contrast between the two functions. If an angle is measured and marked, an angular measure can be produced through a dynamic change that doesn’t represent the measure for the marking. This clearly can be debilitating for students. When looking at digital tools it also becomes obvious that, similar to analog tools, there are always several possibilities for measuring an angle. Then again in contrast to analog tools, digital tools can usually not be used in an equally variable way. In order to still achieve a certain variance in usability good ideas for multi-tools or several different tools are needed. Especially a later change of the definition of an angle does not make a lot of sense for students. Approaches like in GeoGebra or Cinderella, for instance, were the interval of the angular measure can be changed in the preferences (in the interval from 0° to 180° to the interval of 180° to 360°) are surely not being used by students because they simply are not even aware of the possibility. Promising are gesture-based approaches, because several angle definitions can be implemented without the user having to actively decide for them. Then a gesture, which represents an individual understanding, would be enough to trigger the appropriate response. To a certain degree, this is similar to the more analog tools that are also used in regards to the individual understanding of an angle. In general, it should be stated that students predominantly use digital tools that are similar to the ones they would use by hand [17].

5. Design requirements of a digital protractor

As we know learning success in digital learning environments cannot only be achieved through content and pedagogical inclusion, but that software design in particular can have a profound influence [18]. Especially in class time should not be used on learning how to work with a program interface [19] which has already been stated by teachers on several occasions in practice [18]. Furthermore, it is mandatory to evaluate the design aspects of tools respect to the mathematical and mathematic-didactical inclusion of angular measurement. Since this subject is discussed in several fields, like in computer science, mathematics, mathematical didactics, and cognitive psychology it is difficult to account for a comprehensive literature overview. Therefore, the guiding literature for the following part, in particular, will be the works of Mackrell and Kortenkamp/Dohrmann which explicitly deal with the design of tools in dynamic geometrical systems. At crucial points, consideration is also given to the other disciplines.

In the sense of Mackrell, tools for the measurement of angles examined here can be classified as a tool for construction operations. Characteristic for this is that new objects are created using already existing objects. Mackrell also classifies measuring per se in this category. If no new object is created during measurement, but the measurement is displayed directly and not interactively, then the tool is part of the “Information operation”. When applying a constructing tool, this model generally requires four steps ([21], [22]), which will be leading the way for the design of the tool.

1. The selection of the right tool

According to Goldenberg ([23] in [21]), tools can be divided into two categories, regardless of their actual use. On one hand, there are the atomic tools, which cannot be substituted in their function by a combination of other tools. On the other hand, the molecular tools are characterized by the fact that others can replace them by linking. One example of a typical molecular tool is the center point tool. It can simply be replaced with a circle tool and a straight line or line segment tool, if intersections can be marked as such. Molecular tools are not only a shortening of the work process on the drawing surface but according to Mackrell also bring an "affordance" with them. For example, if students also have the tool for marking the center point at their disposal, center point marking is more likely to be included in the design considerations than without. So if a molecular tool is to be added to an already existing canon of tools, it must necessarily have such an enabling character. Otherwise, it would be superfluous despite having a potentially wide range of usage. One of the following questions must, therefore, be answered positively for the inclusion of a new tool:

1. Is the tool atomic?
2. If it is molecular, is its affordance large enough so that a separate tool makes sense?

It is not easy to include the tool developed here into these categories because it fulfills several functions at the same time. Simply measuring an angle could easily be considered atomistic. Certainly, any angle could be approximated using a "straight line with a fixed angle"; however, this way in itself cannot provide a measurement in the same way. The part that is not atomic, however, is the angle mark: essentially, this is just a circular arc that can be easily formed with the appropriate (atomic) tool. So if there is already a "pure" measuring tool, without any marking, which measures the angle in the same way, then our tool is clearly a molecular one. Nevertheless, it offers enormous opportunities in terms of affordance: It uses the typical angle concepts and additionally the typical presentation forms. Thus, it should be more attractive for beginners than the pure simple measuring tool, which usually requires an elaborate angle concept. Kortenkamp and Dohrmann are also in favor of this fit between tool and target group:

> "The decision between designing an easy-to-use interface vs. designing an interface that deliberately enforces intellectual activity of the users by being "difficult" […] is a decision that cannot be independent of the target users, as any intellectual barrier has to be designed appropriately to match the users' experience, prior knowledge, motivation, context, goals, and requirements." ([19], p.60/61)

2. Finding the right tool in the toolbox of the IGS

So why not just provide a separate tool for every angle concept? After all, Geometer Sketchpad also does this – there are five different length measurement tools to cover each of the possible aspects. (According to a personal communication in [21]) However, the commonly held thesis is that too many tools make the surface too complex for it to work with it in a meaningful way. Mackrell, for example, writes:

> "any gain in functionality may well be offset by the resulting increase in complexity" ([21], p.384)

Support for this thesis is easy to find in cognitive psychology: there, a practical approach is the cognitive load theory [24]. It can be summarized in a very superficial way so that a certain cognitive storage quota is available to every student with which they have to economize while learning. Part of this quota (intrinsic cognitive load) must be spent on the learning task itself; this is essentially determined by the difficulty of the task. This requirement certainly cannot be reduced by the IGS itself. Another part has to be "spent" on practicing strategies and recognizing typical situations (germane cognitive load), but also generalizations to larger contexts fall into this area. However, the crucial part, which is also significantly influenced by the design of the geometry systems, is the extrinsic load (extrinsic cognitive load). This is caused by the design of the information to be processed. Now, if a user has to use a lot of capacity to find the right one among a set of tools, there is less capacity left to generalize, for instance, or to match up with other typical situations. The learning effect of each material would simply be reduced in an IGS with too many tools. Thus, from a psychology perspective, it probably does not make sense to provide multiple tools for the same purpose. In a similar theory by Mayer [25] this is also called the coherence principle. Nevertheless, initial results suggest that if such an effect of tool diversity on the cognitive load in IGS exists at all, it is not very profound [26]. However, it is also known in psychology that prior knowledge and intelligence are important factors influencing the cognitive contingent [27], which was not really taken into account in the study by Schimpf and Spannagel.

To simplify the surface, other solutions can be found than the simple omission of functionality. For example, Cinderella divides all the tools into different toolbars – depending on how advanced the user is or what Cinderella is currently being used for, only the presumably needed tools can be displayed. Only the assignment of who and when which toolbar is displayed is tricky. In a school context, for example, the toolbar "School" could be installed as a standard, but a deeper preoccupation with ellipses would then no longer be possible. A dynamic solution that unlocks further elements of the toolbar depending on the level of knowledge would be conceivable (adaptive interface), but can only be implemented on private devices with the same user. However, this situation usually does not apply, especially in school contexts, so that this approach is only useful for individually bound user accounts. This is much easier with the increasing mobile device availability. The approach taken by Cinderella of selecting various toolbars can thus be seen as the most appropriate compromise for schools.

Another important point to easily find the right tool is the grouping within the toolbar. If similar tools are combined in the same area, finding a specific tool could in principle be faster ([26], [21]). However, this assignment is not always clear and depends on the respective understanding of the user. If the user is in the foreground of measuring with our tool, then it should be stored near the other measuring tools, but also a position close to the straight line and line segment tools would be understandable since angles are also associated with straight lines and intersections. If Cinderella is used again as an example, then the angle measuring tool is arranged there with the other measuring tools. The tool to mark the angle, however, seems to have been more of a remnant. It was placed (in the standard toolbar) behind the tool "Define a Function" at the very end of the tool list. Depending on the IGS, pictogram menus are also used (e.g. in Geogebra) that hide a group of tools in a single button. Whatever the philosophy behind the particular geometric program may be, the pictograms for each individual tool should always be meaningful and unambiguous so that they can be found in a way that makes sense ([26], [21]).

3. Usage of the found tool

There are two main approaches to using tools. Either the user first has to click on the tool and then selects the respective objects (Action-Object, or AO), or the selection of the respective objects is followed by the choice of the tool or the action (Object-Action, or OA in [19]). As with the question of tool surfaces, the decision for a procedure is a conceptual one. Thus, two of the first IGS use only one of the paths (Cabri & Sketchpad). More modern implementations, such as in Cinderella, typically use a mixed approach in which both ways lead to the desired design. Sketchometry, on the other hand, as an IGS with innovative gestures and sketches, can not easily be classified in these categories [28]. Beyond these more general considerations, however, the rotational aspect of angles requires that the order of user input is significant. Specifically, this means that the OA approach does not seem feasible ([19], [21]). If an OA approach was followed, the IGS would either have to specify a standardized order of the objects or record the order of object selection even before selecting an activity or tool. This not only seems costly, but the results of such a tool would also be incomprehensible even for users. But when deciding on a tool with AO approach, the question must be asked what happens when a user tries to measure an angle with the OA approach. There are two possible implementations: Either nothing happens at all (or the tool is selected independently of the marking of the points and expects three new points), or an angle is measured. For the reasons mentioned above, the decision on the angle measurement to be measured is not entirely trivial, which is why the OA approach to the tool should not work. However, the problem of "false affordance" is bought in: learners expect a function, which does not happen [29]. This can lead to frustration. Advantages of the AO approach are also the display of tooltips and the previews of the angle marking [21]. With the help of tooltips, the operation can be explained again after the tool has been selected (or also as a mouse-over-text). The emphasis on the order of points will then be made clear again. A preview of the angle marking and the angle measure additionally emphasizes the dynamic character of the tool.

4. Interpretation of the results

For the interpretation of a measurement, in particular, the representation of the measure is significant. Besides the unit itself, decisions about the accuracy of the measure also occur. In terms of various angle aspects, at least degrees, radians, and a number of revolutions should be offered. More than one point behind the comma should not be necessary for the degree measure. Due to smaller amounts in the radiant and rotation display, two decimal places should be offered. In the sense of a practical reuse of marking and measure, these should be clearly depicted in the design description. Helpful in this sense are references to the defining points of the angle marking and the reference to the angle mark for the angle measure.

6. Mathematical realization

The approach of some IGS to realize the drawing plane mathematically with the help of projective geometry to a certain extent also helps with the angular measurement. Without going too much into details (for example, see [30]), a simple calculation formula for angles can be specified. All points have three coordinates (x, y, z) in the projective approach. A point of the drawing plane is assigned to all points of the space that are on the line of origin through the point of the drawing plane. Since the actual Euclidean plane of the drawing is at z = 1, the straight line of origin just described intersects the drawing plane at exactly one point: namely at (x / z, y / z, 1). The interpretation of the homogeneous coordinates as complex vectors (using the vectors I and J) now ensures that the logarithm of the ratio of two vectors corresponds exactly to the angle between the projected vectors.

$Let\ [a,b,c]\ denote\ the\ determinant\ of\ the\ row-vectors\ a,b\ and\ c;$

$Let\ l_\infty\ be\ the\ straight\ line\ at\ infinity\ and\ \ I=(i,-1,0), as\ well\ as\ J=(i,1,0)$

$Let\ \vec{p}\ and\ \vec{q}\ be\ the\ homogeneous\ Coordinates\ of\ the\ planar\ points\ P\ and\ Q,$

$then\ the\ planar\ angle\ \alpha\ between\ these\ two\ points\ is\ given\ by$

$$\alpha=\frac{1}{2i}ln\left(\frac{[\vec{p},I,l_\infty][\vec{q},J,l_\infty]}{[\vec{p},J,l_\infty][\vec{q},I,l_\infty]}\right)$$

Unfortunately, not all angles can be calculated with that. Since with the approach of projective geometry points in the infinite also play a role, it must be considered as well. All lines of origin with z = 0 do not touch the plane of the drawing in the finite, which means that the intersections can be considered very good at points in the infinite on the plane. If such a point at infinity is represented by $(x_0, y_0, 0)$, its line of origin intersects the plane of the drawing, however, in each case in one point in each direction, whereby the point at infinity has a direction fixed to orientation but is not unique. An angle measurement between two points in the finite and a far point as the endpoint is therefore never clear. Apart from this special case, however, an angle can always be specified. However, since the angle defined by points requires more information in order to be able to evaluate the correct orientation of the angle and possibly the number of previous revolutions, further information must be evaluated. The order of the point selection and the tracking of the cursor, however, suffice to clearly define the angle.

7. The concrete tool

The explanations given in the preceding chapters will now be summarized in a concentrated form in order to provide a complete picture of the tool developed here. By way of illustration, a few static images have been added which, as a kind of "storyboard", are intended to adequately represent the dynamic tool.

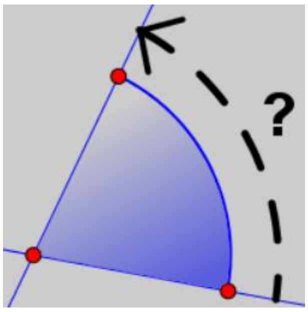


Figure 2:
Icon of the tool

Appearance before using the tool

At the side (fig. 2) the tool icon is shown. It is based on the already existing icons of the tools for angles but adds the new aspect of the filling to the angle marking. The dashed arrow supports the dynamic usage, without explicitly favoring the rotation idea (the arrow is separate from the angle marker and dashed to indicate only the movement). The mouse-over text is "Mark and Measure Angle". When clicking on the tool, the explanatory line adds "Add angle mark and measure by first selecting the vertex and then two further points". Selecting the points first and then the tool produces no result (no OA approach).

Measuring with the new tool

The initial situations for angular measurements are usually sections of straight lines or rays. The operation is also shown in Figure 3 in the steps described below. As already explained above, the tool should successively select vertices, the point to be rotated and finally the endpoint. After the vertex is selected, the mouse pointer should already have a transparent circle attached (similar to the circle-around-center tool, shown in Figure 3) to demonstrate a variable mark size. After determining the point to be rotated (2), the mouse movement decides on the direction of the rotation angle. Dynamically, the angle mark is also displayed, which is filled with a gradual transparency to identify the first and second side (3). Depending on the mouse movement results a negative (3) or positive-oriented angle (5). Even obtuse angles can be detected (4). A final click on the endpoint sets the angle clearly (6). The size of the angle marking can now be changed dynamically with the movement of the second point.

The measure of the angle and the orientation of the current angle are displayed in the upper right corner of the drawing area. In the design description, the angle mark and the angle measure appear separately, as shown here (Fig. 4). Nevertheless, the angle and the angle mark coincide at all times, since the representation of the angle marking depends on the calculation of the measure. The display of the measure, however, can be removed independently of the drawing area.

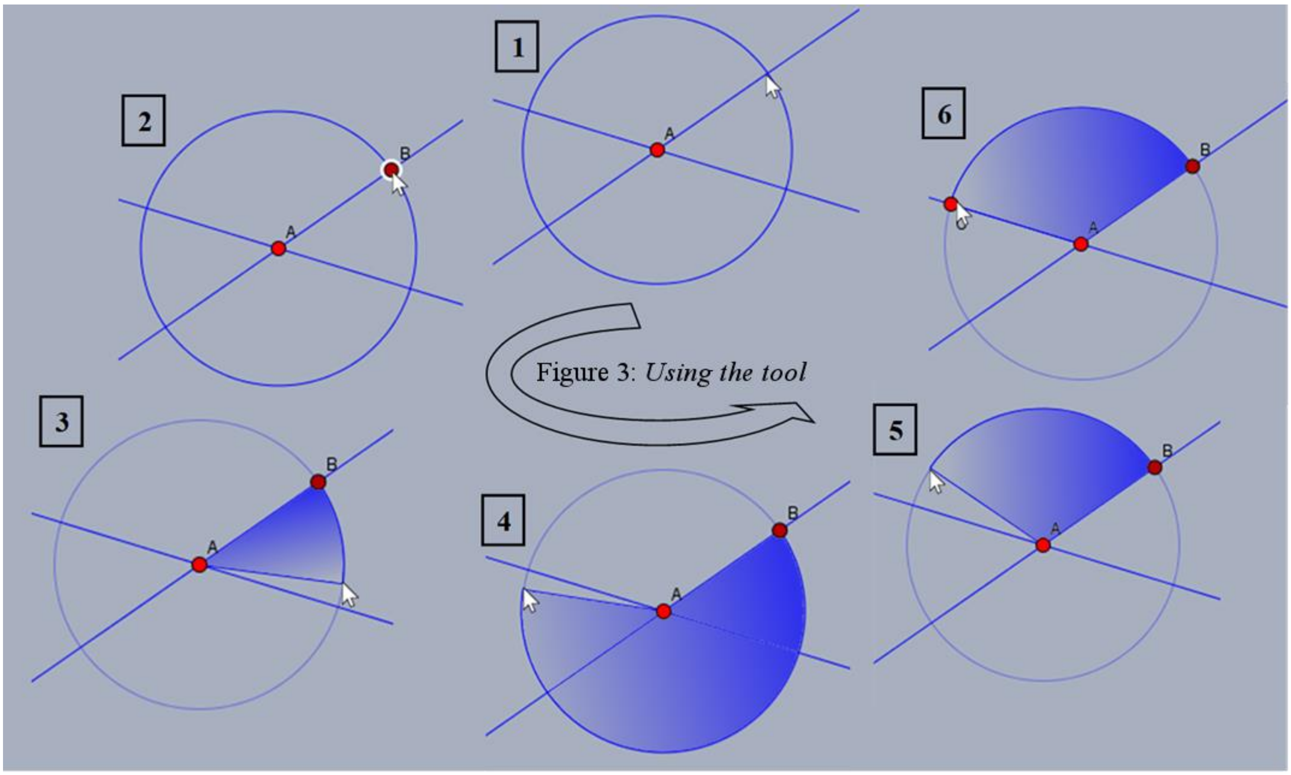


Figure 3: *Using the tool*

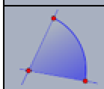
	Who?	What?	Where?
	C1	Angle Marking	Vertex: A
	Text0	Angle Measure	C1

Figure 4: *Construction text of the objects*

In the special case of over-rotations, in the second variant of the tool, the angle marking changes from the circle segment to a spiral, as shown in Figure 5. Angles can then be displayed beyond the full angle. In addition to measurement and orientation, the number of complete revolutions is now also available as an output. Even with the spirals, the size of the mark can still be changed by changing the second point.

Measurement and orientation are generally displayed as soon as the point to be rotated has been determined with the mouse. The first displayed dimension is then 0 °, although no orientation can be determined here. Figure 6 shows an example of the displayed dimensions of the previous German versions of the tool. It is noticeable that the orientation is not expressed in the measure itself by a sign.

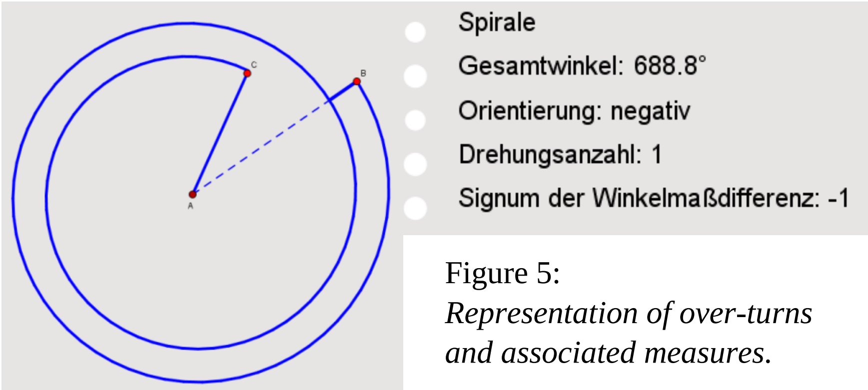


Figure 5:
Representation of over-turns and associated measures.

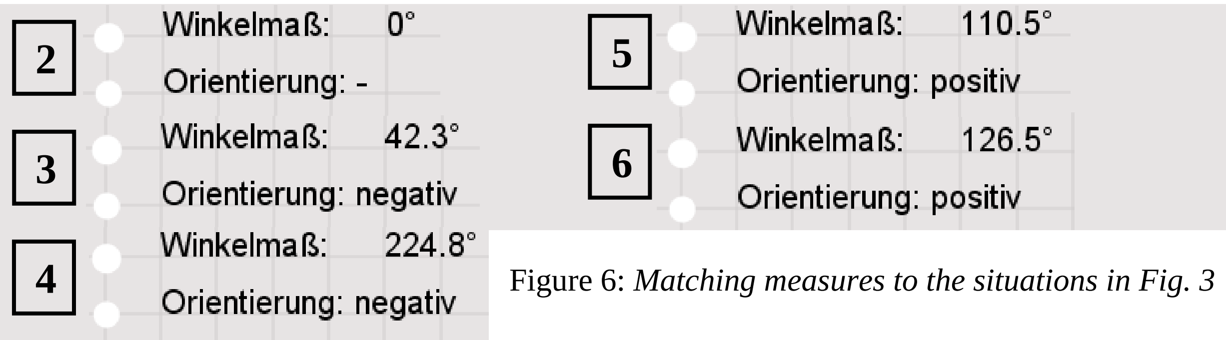


Figure 6: *Matching measures to the situations in Fig. 3*

For the special case of points at infinity, a distinction must be made between the two variants of the tool. In the variant without overrotation, if the vertex is moved to infinity, or is there at the beginning, the measure should be 0 °. In particular, since the angle-forming lines are then parallel, that also makes sense. An angle mark can then not be displayed in the planar geometry and should therefore not be taken into account. If such a vertex is now shifted from the opposite direction back to the finite plane, the angle should jump against the usual Cinderella philosophy. The reason for this is the support of the angle concept depending on rays. The definition of the angle marking with three points also speaks for this jumping, otherwise, the angle marking could suddenly be on the opposite site of the actual starting point. The orientation of the angle can be easily taken from the previous (finite) situation. In the case of over-rotations, consideration of these cases becomes too complex. There is no more sensible decision possible for the number of turns. Accordingly, in the variant with over-turns no mark and no measure should be displayed when the vertex is moved to infinity. For the sake of simplicity, moreover, the number of revolutions should be taken over in advance when the point is shifted back to the finite. Otherwise, the angle segment can jump, as in the first variant of the tool.

8. Conclusion

This article explored how certain aspects of digital angle tools can be helpful in developing a comprehensive understanding of the term angle. In particular, the rotational approach proved to be underrepresented in Germany, although it is potentially helpful against many misconceptions. In the course of this, a concrete tool including the ideas for its implementation was developed. This tool has been optimized with design considerations in mind but is certainly not meant to be an ultima ratio. The functional prototype was developed in the IGS Cinderella and can be accessed at https://bit.ly/2XZrjga (no password needed). There the mathematical embedding was not used in a direct form. Instead of the cross-ratio of determinants, the arctan2 was used. However, this does not change the other mode of operation in the flat geometry. In addition to the angular measurement and the orientation, further information on implementation is also presented. In the current form, a gradual transparency of the angle mark could not yet be realized because parametric functions in Cinderella cannot yet be filled with color. Some of the decisions made, especially those of the tool operations, are explicitly bound to the philosophy of Cinderella and are therefore not clear in these perspectives (points at infinity, AO / OA approach, ...). In another IGS, therefore, an angle-measuring tool should be rethought again, even if the one presented here can certainly represent a good first approach. In addition to this product, this article has above all emphasized how important and comprehensive consideration should be given to angle tools in IGS to be useful for schools. Therefore, in the future, an intensive examination of angular understanding of students must be the claim of modern IGS. The dynamics of geometry programs will also be more important due to the increasing availability of personal touch devices. This means that even existing tools must always be adapted to the current situation in their operation.

9. References

[1] Krainer, Konrad (1990): *Lebendige Geometrie. Überlegungen zu einem integrativen Verständnis von Geometrieunterricht anhand des Winkelbegriffes.* Zugl.: Klagenfurt, Univ., Diss., 1989. Frankfurt am Main: Lang (Europäische Hochschulschriften Reihe 11, Pädagogik, 409).

[2] Etzold, Heiko (2017): *Winkel aus der Sicht von Informationen.* In: Andreas Filler und Anselm Lambert (Hg.): Von Phänomenen zu Begriffen und Strukturen -Konkrete Lernsituationen für den Geometrieunterricht. Vorträge auf der 32. Herbsttagung des Arbeitskreises Geometrie in der Gesellschaft für Didaktik der Mathematik vom 11. bis 13. September 2015 und auf der 33. Herbsttagung vom 09. bis 11. September 2016 in Saarbrücken. 1. Auflage (AK Geometrie), p. 35–44.

[3] Close, Gillian Susan (1982): *Children's understanding of angle at the primary, secondary transfer stage.* Zugl.: London, Council for National Acad. Awards, Diss., 1981. London: Dep. of Mathematical Sciences and Computing Polythechnic of the South Bank.

[4] Dohrmann, Christian; Kuzle, Ana (2015): *Winkel in der Sekundarstufe I – Schülervorstellungen erforschen.* In: Matthias Ludwig, Andreas Filler und Anselm Lambert (Hg.): Geometrie zwischen Grundbegriffen und Grundvorstellungen. Jubiläumsband des Arbeitskreises Geometrie in der Gesellschaft für Didaktik der Mathematik. Wiesbaden: Springer Spektrum, p. 29–42.

[5] Krainer, Konrad (1991): *Consequences of a Low Level of Acting and Reflecting in Geometry Learning -Findings of Interviews on the Concept of Angle.* In: Fulvia Furinghetti (Hg.): Proceedings Fifteenth PME Conference. Assisi (Italy) June 29 -July 4. S.l.: Program Committee of the 15th PME Conference, p. 254–261.

[6] Freudenthal, Hans (1977): *Mathematik als pädagogische Aufgabe.* 2., durchges. Aufl. Stuttgart: Klett (Klett-Studienbücher Klett-Studienbücher Mathematik).

[7] Wilson, Patricia S.; Adams, Verna M. (1992): *A Dynamic Way To Teach Angle And Angle Measure.* In: The Arithmetic Teacher 39 (39), p. 6–13.

[8] Olson, Melfried; Zenigami, Fay; Okazaki, Claire (2008): *take time for action: Students' Geometric Thinking about Rotations and Benchmark Angles.* In: Mathematics Teaching in the Middle School 14 (1), p. 24–26.

[9] Piaget, Jean; Inhelder, Bärbel (1999): *Die Entwicklung des räumlichen Denkens beim Kinde.* 3. Aufl. Stuttgart: Klett-Cotta (Gesammelte Werke, Studienausgabe / Jean Piaget. Mit e. Einf. von Hans Aebli ; Bd. 6).

[10] Mitchelmore, Michael C. (1998): *Young Students' Concepts of Turning and Angle.* In: Cognition and Instruction 16 (3), S. 265–284. DOI: 10.1207/s1532690xci1603_2.

[11] Mitchelmore, Michael; White, Paul (1998*): Development of angle concepts: A framework for research.* In: Mathematics Education Research Journal 10 (3), p. 4–27. DOI: 10.1007/BF03217055

[12] Dohrmann,Christian; Kuzle, Ana (2013a): *Begriffsbildung im Mathematikunterricht der SEK I zum Thema Winkel.* In: Andreas Filler und Matthias Ludwig (Hg.): Wege zur Begriffsbildung für den Geometrieunterricht. Ziele und Visionen 2020 ; Vorträge auf der 29. Herbsttagung des Arbeitskreises Geometrie in der Gesellschaft für Didaktik der Mathematik vom 14. bis 16. September 2012 in Saarbrücken. 1. Ed. Hildesheim: Franzbecker, p. 63–72.

[13] Weth, Thomas; Richter-Gebert, Jürgen; Kortenkamp, Ulrich (Hg.) (2002): *Dynamische Geometrie: Grundlagen und Möglichkeiten.* Online available at http://www.cinderella.de/papers/DG_GM.pdf.

[14] Vom Hofe, Rudolf (1992): *Grundvorstellungen mathematischer Inhalte als didaktisches Modell.* In: JMD 13 (4), p. 345–364. DOI: 10.1007/BF03338785.

[15] Dohrmann, Christian; Ana Kuzle; Jürgen Roth; J. Ames (2014): *Begriffsbildung zum Winkel in der Sek I: Auf der Suche nach Grundvorstellungen.* In: 48. Jahrestagung der Gesellschaft für Didaktik der Mathematik.

[16] Dohrmann, Christian; Kuzle, Ana (2013b): *Past-Present-Future: Winkel anschaulich unterrichten!* In: Gilbert Greefrath (Hg.): Beiträge zum Mathematikunterricht 2013. Vorträge auf der 47. Tagung für Didaktik der Mathematik ; Jahrestagung der Gesellschaft für Didaktik der Mathematik vom 4.3.2013 bis 8.3.2013 in Münster, Bd.1. Münster, Dortmund: WTM Verl. für Wiss. Texte u. Medien; IEEM Institut für Entwicklung und Erforschung des Mathematikunterrichts, p. 260–263.

[17] Ladel, Silke; Knopf, Julia; Weinberger, Armin (Hg.) (2018): *Digitalisierung und Bildung.* Wiesbaden: Springer Fachmedien Wiesbaden. Online available at http://dx.doi.org/10.1007/978-3-658-18333-2.57

[18] Laborde, Colette (2007): *The role and uses of technologies in mathematics classrooms. Between challenge and modus Vivendi.* In: Canadian Journal of Science, Mathematics and Technology Education 7 (1), p. 68–92. DOI: 10.1080/14926150709556721.

[19] Kortenkamp, Ulrich H.; Dohrmann, Christian (2010): *User Interface Design For Dynamic Geometry Software.* In: Acta Didactica Napocensia 3 (2), S. 59–66. Online verfügbar unter https://files.eric.ed.gov/fulltext/EJ1056067.pdf

[20] Ruthven, Kenneth; Hennessy, Sara; Deaney, Rosemary (2005): *Current practice in using dynamic geometry to teach about angle properties.* In: Micromath 21 (1), p. 9–13.

[21] Mackrell, Kate (2011): *Design decisions in interactive geometry software.* In: ZDM Mathematics Education 43 (3), p. 373–387. DOI: 10.1007/s11858-011-0327-4.

[22] Sedig, Kamran; Sumner, Mark (2006): *Characterizing Interaction with Visual Mathematical Representations.* In: Int J Comput Math Learning 11 (1), p. 1–55. DOI: 10.1007/s10758-006-0001-z.

[23] Goldenberg, E. P.; Scher, D.; Feurzeig, N. (2008): *What Lies Behind Dynamic Interactive Geometry Software?* In: Glendon W. Blume und Mary Kathleen Heid (Hg.): Research on Technology and the Teaching and Learning of Mathematics: Vol. 2. Cases and Perspectives. Charlotte, N.C, Reston, Va.: National Council of Teachers of Mathematics (Research on technology and the teaching and learning of mathematics, v. 2), p. 53–88

[24] Sweller, John (1988): *Cognitive Load During Problem Solving. Effects on Learning.* In: Cognitive Science 12 (2), p. 257–285. DOI: 10.1207/s15516709cog1202_4.

[25] Mayer, Richard E. (2009): *Multimedia learning.* Second edition. Cambridge: Cambridge University Press. Online available at https://doi.org/10.1017/CBO9780511811678.

[26] Schimpf, Florian; Spannagel, Christian (2011): *Reducing the graphical user interface of a dynamic geometry system.* In: ZDM Mathematics Education 43 (3), p. 389–397. DOI: 10.1007/s11858-011-0325-6.

[27] Lee, Hyunjeong; Plass, Jan L.; Homer, Bruce D. (2006): *Optimizing cognitive load for learning from computer-based science simulations.* In: Journal of Educational Psychology 98 (4), p. 902–913. DOI: 10.1037/0022-0663.98.4.902.

[28] Gerhäuser, Michael (2014): Interaktive Mathematik im Browser. Werkzeuge, Bibliotheken und Programmierumgebungen. Dissertation. Universität Bayreuth, Bayreuth. Fakultät für Mathematik, Physik und Informatik. Online available at https://epub.uni-bayreuth.de/1913/1/dissertation.pdf

[29] Gaver, William W.: *Technology affordances.* In: Scott P. Robertson, Gary M. Olson und Judith S. Olson (Hg.): the SIGCHI conference. New Orleans, Louisiana, United States, p. 79–84.

[30] Richter-Gebert, Jürgen; Orendt, Thorsten (Hg.) (2009): *Geometriekalküle.* Berlin, Heidelberg: Springer Berlin Heidelberg (Springer-Lehrbuch).